[6]Kane, T. R., "Real Solutions of Sets of Nonlinear Equations," *AIAA Journal*, Vol. 4, No. 10, 1966, p. 1880.
[7]Kane, T. R., and Levinson, D. A., *Autolev Users' Manual*, Online Dynamics, Inc., Stanford, CA, 1999, p. 147.

Optimal Control Bolza and Transformed Mayer Problems with Feedback Linearized State Equations

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I. Introduction

MINIMUM time solution of classes of linear and nonlinear dynamic systems with bounded inputs is well known to be bang–bang, that is, the optimal inputs switch between minimum and maximum values.[1–3] Structures of their solution are well documented in the literature. Mayer problems for classes of feedback linearizable nonlinear systems with mixed input and state constraints also have special structures: At least one constraint is always on the boundary.[4] This result is a generalization of the classical bang–bang result for linear systems.

In this paper, we consider Bolza optimal control problems where the dynamic systems are given by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $x \in \mathcal{R}^n$ are the states, $u(t) \in \mathcal{R}^m$ are the controls, and the matrix $g(x)$ consists of smooth vector fields $g_1(x), \ldots, g_m(x)$. The objective is to

$$\text{minimize } J = \Phi[x(t_f), t_f] + \int_{t_0}^{t_f} F(x, u)\, \mathrm{d}t \tag{2}$$

where $t \in [t_0, t_f]$ denotes the time. The scalar functions $\Phi[x(t_f), t_f]$ and $F(x, u)$ have continuous first and second partial derivatives with respect to their arguments. The inequality constraints are $s(x, u) \leq 0$ and $c(x) \leq 0$. The terminal states satisfy $\psi[x(t_0), x(t_f)] = 0$ with $\partial\Phi/\partial t_f \neq 0$.

A Mayer optimal control problem does not have an explicit integral in its cost: This is a major difference between Bolza and Mayer optimal control problems. As is well known, every Bolza problem can be converted to a Mayer problem through the following two steps: An auxiliary state x_{n+1} is added to Eq. (1) with the dynamics $\dot{x}_{n+1} = F(x, u)$, thereby, modifying the state equations to

$$\dot{x} = f(x) + g(x)u, \qquad \dot{x}_{n+1} = F(x, u) \tag{3}$$

The cost in Eq. (2) can now be rewritten as

$$\text{minimize } J = \Phi[x(t_f), t_f] + x_{n+1}(t_f) \tag{4}$$

with $x_{n+1}(t_0) = 0$. In this new form of the problem, the inequality constraints and boundary constraints do not change.

As outlined in Ref. 4, special structures are present in the optimal solution of a Mayer problem if the governing system dynamics are feedback linearizable, that is, diffeomorphic to chains of integrators. Such structures will also be present in the solution of a Bolza problem if the set of Eqs. (3) is feedback linearizable. This motivates the statement of the problem discussed in this Note: given the state Eqs. (1) and the integrand $F(x, u)$ in Eq. (2), what forms of $F(x, u)$ will make the augmented state Eqs. (3) feedback linearizable?

The organization of this Note is as follows: Section II derives the necessary conditions for Eqs. (3) to be feedback linearizable. Consistent with these necessary conditions, linear and quadratic forms of $F(x, u)$ are studied in detail in Secs. III and IV, respectively. The results of this paper are illustrated by an example in Sec. V.

II. Conditions on $F(x, u)$

A. Background Results

Definition 1: A system $\dot{x} = f(x, u)$, $x \in R^n$, $u \in R^m$ is said to be differentially flat if there exists $y \in R^m$ dependent on x, u, and their derivatives up to a finite order such that x and u can be written as functions of y and its derivatives up to a finite order.

Flat systems are dynamic feedback linearizable.[5,6] Necessary and sufficient conditions do not exist that guarantee a general system to be flat. However, there are results for special classes of systems. For example, flatness is equivalent to controllability for linear systems. Other systems that are known to be flat are static feedback linearizable systems, which are characterized as follows[7,8]:

Theorem 1: A control affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

is static feedback linearizable if and only if the following distributions are constant rank and involutive:

$$D_0 = \langle g_1, \ldots, g_m \rangle$$

$$D_i = \left\langle D_{i-1}, ad_f^i g_1, \ldots, ad_f^i g_m \right\rangle, \qquad \forall i = 1, \ldots, n-1$$

and D_{n-1} spans R^n.

Proposition 1: A single-input system is dynamic feedback linearizable if and only if it is static feedback linearizable. The proof can be found in several papers and textbooks, for example, Refs. 6 and 9.

Proposition 2: A necessary condition for a system

$$\dot{x} = f(x, u), \qquad u \in R^m \tag{5}$$

$$\dot{x}_{n+1} = F(x, u) \tag{6}$$

to be dynamic feedback linearizable is that Eq. (5) be dynamic feedback linearizable.

Proof: A dynamic compensator for Eq. (5) is

$$\dot{z} = a(x, z, v), \qquad u = b(x, z, v) \tag{7}$$

Therefore, if

$$z = x_{n+1}, \qquad F(x, u) = a(x, z, v), \qquad u = v$$

we can view Eq. (6) to be a dynamic compensator for Eq. (5). Then, the dynamic feedback linearizability of Eqs. (5) and (6) implies dynamic feedback linearizability of Eq. (5). That is, Eq. (5) is dynamic feedback linearizable. □

Corollary 1: If Eq. (5) is restricted to a single input and the overall system is dynamic feedback linearizable, then Eq. (5) is static feedback linearizable.

Proof: It is clear from the preceding proposition and equivalence between static and dynamic feedback linearization for single-input systems.

Corollary 2: If Eqs. (5) and (6) are linear and the overall system is controllable, then Eq. (5) is controllable.

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Remark: Because there does not exist a necessary and sufficient condition to know whether or not a system is dynamic feedback linearizable, in the sequel, when dealing with nonlinear systems, we restrict ourselves to static feedback linearizable systems.

B. Necessary Condition

Proposition 3: If the system

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j \tag{8}$$

$$\dot{x}_{n+1} = F(x, u_1, \ldots, u_m) \tag{9}$$

is static feedback linearizable, then

$$\frac{\partial^2 F}{\partial u_i \partial u_j} = 0, \qquad \forall i, j \mid 1 \le i \le n \qquad 1 \le j \le n$$

Proof: Equations (8) and (9) can be rewritten in the affine form by introducing $\forall j = 1, \ldots, m$, the new state variables $y_j = u_j$, the new controls v_j, and the new equations $\dot{y}_j = v_j$:

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) y_j, \qquad \dot{x}_{n+1} = F(x, y_1, \ldots, y_m)$$

$$\dot{y}_j = v_j, \qquad \forall j = 1, \ldots, m$$

A straightforward computation shows that the second distribution associated with this system is static feedback linearizable if and only if $\partial^2 F / \partial y_i \partial y_j = 0$. Hence, in the original variables, $\partial^2 F / \partial u_i \partial u_j = 0$. □

Therefore, the only allowable forms for

$$F(x, u_1, \ldots, u_m) = l(x) + \sum_{j=1}^{m} l_j(x) u_j$$

III. Admissible Linear Costs

In this section, we consider both necessary and sufficient conditions for feedback linearizability of Eqs. (8) and (9). We make the following assumptions: 1) The system described by Eq. (8) is feedback linearizable and has a Brunovsky canonical form. 2) $F(x, u)$ is linear in both states and inputs, which is a special case of the form required in Proposition 3. Let the Brunovsky canonical form for Eq. (8) be

$$\dot{y}_i = y_{i+1}, \qquad \forall i = 1, \ldots, n \qquad i \ne l_j$$

$$\dot{y}_{l_j} = v_j, \qquad l_j = \sum_{r=1}^{j} k_r \qquad j = 1, \ldots, m$$

$$\dot{y}_{n+1} = \sum_{i=1}^{n} \lambda_i y_i + \sum_{j=1}^{m} \mu_j v_j \tag{10}$$

where k_j are the controllability indices (e.g., see Ref. 8 for a description of these indices).

Proposition 4: System (10) is controllable if and only if at least one of the coefficients $\{\lambda_1, \lambda_{l_1+1}, \ldots, \lambda_{l_{n-1}+1}\}$ is different from zero.

Proof: It can be shown that all of the state variables and the inputs can be written as linear functions of the variables $w_1, \ldots, w_m$ and their derivatives, where

$$w_j = y_{l_j+1}, \qquad \forall j = 1, \ldots, m-1$$

$$w_m = y_{n+1} - \sum_{i=1, i \ne l_j}^{n} \lambda_{i+1} y_i - \sum_{j=1}^{m} \mu_j y_{l_j} \tag{11}$$

Hence, the system is flat and, therefore, controllable. This finishes the proof of the sufficiency.

To see the necessity, define again w_m as in Eq. (11). If all of the coefficients $\{\lambda_1, \lambda_{l_1+1}, \ldots, \lambda_{l_{n-1}+1}\}$ are zero, then $\dot{w}_m = 0$. Therefore, the system is not controllable as proven in Ref. 6. □

IV. Admissible Quadratic Costs

In this section, we consider both necessary and sufficient conditions for feedback linearizability of Eqs. (8) and (9) under the following assumptions: 1) The system described by Eq. (8) is feedback linearizable and has a Brunovsky canonical form. 2) $F(x, u)$ is quadratic in states but only linear in the inputs, consistent with Proposition 3. Let the quadratic cost have the following form:

$$F(x, u_1, \ldots, u_m) = \sum_{i,j=1}^{n} a_{i,j} x_i x_j + \sum_{i=1}^{n} b_i x_i + \sum_{j=1}^{m} \left[\sum_{i=1}^{n} c_i^j x_i + d_j \right] u_j \tag{12}$$

The following theorem specifies further restrictions on admissible $F(x, u_1, \ldots, u_m)$.

Theorem 2: The system in Brunovsky form, augmented by an additional equation to reflect the cost in Eq. (12), is

$$\dot{x}_i = x_{i+1}, \qquad \forall i = 1, \ldots, n \qquad i \ne l_j$$

$$\dot{x}_{l_j} = u_j, \qquad l_j = \sum_{r=1}^{j} k_r \qquad j = 1, \ldots, m$$

$$\dot{x}_{n+1} = \sum_{i,j=1}^{n} a_{i,j} x_i x_j + \sum_{i=1}^{n} b_i x_i + \sum_{j=1}^{m} \left[\sum_{i=1}^{n} c_i^j x_i + d_j \right] u_j \tag{13}$$

where $k_1 \ge k_2 \ge \cdots \ge k_m$ are the controllability indices (for a description on how to compute these indices, see, for example, Ref. 8). This augmented system is static feedback linearizable if and only if

$$\sum_{s=1}^{r-1} (-1)^{s+1} \left(a_{l_j - r - s + 1, l_k - r + s + 1} + a_{l_k - r + s + 1, l_j - r - s + 1} \right) - \sum_{s=1}^{r} (-1)^{s+1} \left(a_{l_k - r - s + 2, l_j - r + s} + a_{l_j - r + s, l_k - r - s + 2} \right) = (-1)^r \left(c^k_{l_j - 2r + 1} + c^j_{l_k - 2r + 1} \right)$$

$$\forall j, k = 1, \ldots, m \qquad \forall r = 0, \ldots, k_m - 1$$

$$\sum_{s=1}^{r} (-1)^{s+1} \left(a_{l_j - r - s + 1, l_k - r + s} + a_{l_k - r + s, l_j - r - s + 1} \right) - \sum_{s=1}^{r} (-1)^{s+1} \left(a_{l_k - r - s + 1, l_j - r + s} + a_{l_j - r + s, l_k - r - s + 1} \right) = (-1)^{r-1} \left(c^k_{l_j - 2r} - c^j_{l_k - 2r} \right)$$

$$\forall j, k = 1, \ldots, m \qquad \forall r = 0, \ldots, k_m - 1$$

Here, $a_{l_k - \alpha, l_j - \beta} = 0$ if $l_k - \alpha \le l_{k-1}$ or $l_j - \beta \le l_{j-1}$. Similarly, $c^k_{l_j - \alpha} = 0$ (respectively $c^j_{l_k - \beta} = 0$) if $l_j - \alpha \le l_j - 1$ (respectively $l_k - \beta \le l_{k-1}$).

The proof of Theorem 2 can be found in the Appendix.

V. Example

The dynamic equations of motion for a flexible joint link are given in Ref. 4

$$I\ddot{q}_1 + MgL\sin(q_1) + k(q_1 - q_2) = 0, \qquad J\ddot{q}_2 - k(q_1 - q_2) = u \tag{14}$$

Choosing the state variables $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, and $x_4 = \dot{q}_2$, the state equations are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -(MgL/I)\sin(x_1) - (k/I)(x_1 - x_3)$$
$$\dot{x}_3 = x_4, \qquad \dot{x}_4 = (k/J)(x_1 - x_3) + (1/J)u \tag{15}$$

A globally diffeomorphic transformation and a corresponding feedback law transforms the system into Brunovsky form[10]:

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_3, \qquad \dot{y}_3 = y_4, \qquad \dot{y}_4 = v \tag{16}$$

The inverse transformations $x_i = x_i(y_1, y_2, y_3, y_4)$ and $u = u(y_1, y_2, y_3, y_4, v)$ are also available in Ref. 10.

According to Proposition 4, admissible linear cost in the new variables

$$F(y, v) = \sum_{i=1}^{4} \lambda_i y_i + \mu v, \qquad \lambda_1 \neq 0$$

will transform the augmented system to Brunovsky's form. For the choice $\lambda_1 = 1$, these equations are

$$\dot{z}_i = z_{i+1}, \qquad \forall i = 1, \ldots, 4 \qquad \dot{z}_5 = v \tag{17}$$

The transformations according to Eq. (11) are

$$z_1 = y_5 - \lambda_2 y_1 - \lambda_3 y_2 - \lambda_4 y_3 - \mu y_4$$
$$z_i = y_{i-1}, \qquad \forall i = 2, \ldots, 5 \tag{18}$$

Similarly, according to Theorem 2, admissible quadratic costs in the new variables have the form

$$F(y, v) = \sum_{i,j=1}^{4} a_{i,j} y_i y_j + \sum_{i=1}^{4} b_i y_i + \sum_{i=1}^{4} c_i y_i v + dv \tag{19}$$

where $a_{4,4} = c_3$, $a_{3,3} - a_{2,4} - a_{4,2} = c_1$, and $a_{1,3} + a_{3,1} = a_{2,2}$. These costs can also be mapped to the original states and input variables using the inverse transformation and feedback law. For example, a cost in the original variables of the form

$$F(x_1, x_2) = \sum_{i,j=1}^{2} a_{i,j} x_i x_j + \sum_{i=1}^{2} b_i x_i \tag{20}$$

satisfies all of the conditions for a Mayer's problem with state equations in Brunovsky form if $a_{2,2} = 0$.

For illustration, we choose a Bolza optimal control problem as

$$\text{minimize } J = t_f + \int_0^{t_f} [y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \lambda_4 y_4 + \mu v]\,\mathrm{d}t \tag{21}$$

subject to Eq. (15) and inequality constraints $|v| \leq 1$. Note that to use the structure of the results, we write the cost in the transformed variables $y_1, \ldots, y_4$, v, and $\lambda_2, \ldots, \lambda_4$ and μ are arbitrarily selected constants. The boundary conditions are taken as $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $x_3(0) = x_{30}$, $x_4(0) = x_{40}$, $x_1(t_f) = x_{1f}$, $x_2(t_f) = x_{2f}$, $x_3(t_f) = x_{3f}$, and $x_4(t_f) = x_{4f}$.

When these results are used, the transformed Mayer optimal control problem is

$$\text{minimize } J = t_f + y_5(t_f) = z_{1f} + \lambda_2 z_{2f} + \lambda_3 z_{3f} + \lambda_4 z_{4f} + \mu z_{5f} \tag{22}$$

subject to Eqs. (17) and (18) and inequality constraints $|v| \leq 1$. The boundary conditions for the new problem are $z_1(0) = z_{10}$, $z_2(0) = z_{20}$, $z_3(0) = z_{30}$, $z_4(0) = z_{40}$, $z_5(0) = z_{50}$, $z_2(t_f) = z_{2f}$, $z_3(t_f) = z_{3f}$, $z_4(t_f) = z_{4f}$, $z_5(t_f) = z_{5f}$ and are obtained by transforming the given boundary conditions, with $z_1(t_f)$ free. The solution of this new Mayer problem can be obtained using standard solution techniques.[1]

VI. Conclusions

It was shown that classes of optimal control Bolza problems can be transformed to Mayer problems with feedback linearized state equations. It was proved that the integrand in the cost functional must necessarily be linear in the control inputs. Furthermore, both necessary and sufficient conditions were derived for such integrands to be transformed to a Mayer problem with the Brunovsky form of the state equations. Also, the results were illustrated by an example. We believe that this is an alternate approach to solve Bolza optimal control problems by transforming them to Mayer problems with special structure of the state equations.

Appendix: Proof of Theorem 2

Let

$$f(x) = \sum_{\substack{i=1\\ i\neq l_j}}^{i=n} x_{i+1}\frac{\partial}{\partial x_i} + \left(\sum_{i,j=1}^{n} a_{i,j}x_i x_j + \sum_{i=1}^{n} b_i x_i\right)\frac{\partial}{\partial x_{n+1}}$$

be the drift vector field, and let

$$g_j = \frac{\partial}{\partial x_{l_j}} + \left[\sum_{i=1}^{n}\left(c_i^j x_i\right) + d_j\right]\frac{\partial}{\partial x_{n+1}}$$

be the vector field associated with inputs of the system $\forall j = 1, \ldots, m$.

It is easy to see by induction that the distributions associated with the system are

$$D_0 = \left\langle\left\{\eta_0^j = g_j,\ j = 1, \ldots, m\right\}\right\rangle$$
$$D_r = \left\langle\left\{\eta_p^j,\ j = 1, \ldots, m,\ p = 0, \ldots, r\right\}\right\rangle$$

where

$$\eta_p^j = \frac{\partial}{\partial x_{l_j - p}} + \left[b_{l_j - r + 1} + \sum_{s=1}^{s=p}(-1)^{s+1}\sum_{\substack{i=1\\ i\neq l_j}}^{i=n}\left(a_{i,l_j - l + s} + a_{l_j - l + s,i}\right)\right.$$
$$\left.\times\ x_{i+s-1} + (-1)^p\sum_{\substack{i=1\\ i\neq l_j}}^{i=n} c_i^j x_{i+p}\right]\frac{\partial}{\partial x_{n+1}}$$

Therefore, assuming involutivity of $D_0, \ldots, D_{r-1}$, the involutivity conditions of D_r read as follows:

$$\left[\eta_r^j, \eta_p^k\right] \in D_r, \qquad \forall 0 \leq p \leq r \qquad \forall j, k = 1, \ldots, m$$

Since

$$\left[\eta_r^j, \eta_p^k\right] = \left[\eta_{r-1}^j, \eta_{p+1}^k\right] \in D_{r-1} \subset D_r$$
$$\forall 0 \leq p \leq r-2 \qquad \forall j, k = 1, \ldots, m$$

the only conditions to be checked for the distribution D_r are

$$\left[\eta_r^j, \eta_{r-1}^k\right] \in D_r, \qquad \left[\eta_r^j, \eta_r^k\right] \in D_r$$

On evaluating, it can be shown that

$$\left[\eta_r^j, \eta_{r-1}^k\right] = \left[\sum_{s=1}^{r-1}(-1)^{s+1}\left(a_{l_j - r - s + 1, l_k - r + s + 1}\right.\right.$$
$$\left. + a_{l_k - r + s + 1, l_j - r - s + 1}\right) - \sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k - r - s + 2, l_j - r + s}\right.$$
$$\left.\left. + a_{l_j - r + s, l_k - r - s + 2}\right) + (-1)^{r-1}\left(c_{l_j - 2r + 1}^k + c_{l_k - 2r + 1}\right)\right]\frac{\partial}{\partial x_{n+1}}$$

Since $\partial/\partial x_{n+1} \notin D_r$,

$$\sum_{s=1}^{r-1}(-1)^{s+1}\left(a_{l_j-r-s+1,l_k-r+s+1}+a_{l_k-r+s+1,l_j-r-s+1}\right)$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+2,l_j-r+s}+a_{l_j-r+s,l_k-r-s+2}\right)$$

$$=(-1)^r\left(c^k_{l_j-2r+1}+c^j_{l_k-2r+1}\right)$$

$$\forall j,k=1,\ldots,m \qquad \forall r=0,\ldots,k_m-1$$

On the other hand,

$$\left[\eta_r^j,\eta_r^k\right]=\left[\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_j-r-s+1,l_k-r+s}+a_{l_k-r+s,l_j-r-s+1}\right)\right.$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+1,l_j-r+s}+a_{l_j-r+s,l_k-r-s+1}\right)$$

$$\left.+(-1)^r\left(c^k_{l_j-2r}-c_{l_k-2r}\right)\right]\frac{\partial}{\partial x_{n+1}}$$

This implies

$$\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_j-r-s+1,l_k-r+s}+a_{l_k-r+s,l_j-r-s+1}\right)$$

$$-\sum_{s=1}^{r}(-1)^{s+1}\left(a_{l_k-r-s+1,l_j-r+s}+a_{l_j-r+s,l_k-r-s+1}\right)$$

$$=(-1)^{r-1}\left(c^k_{l_j-2r}-c_{l_k-2r}\right)$$

$$\forall j,k=1,\ldots,m \qquad \forall r=0,\ldots,k_m-1 \qquad \square$$

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References

[1]Athans, M., and Falb, P. L., *Optimal Control*, McGraw–Hill, New York, 1966, pp. 200–350.

[2]Kirk, D. E., *Optimal Control Theory: An Introduction*, Electrical Engineering Ser., Prentice–Hall, Engelwood Cliffs, NJ, 1970, pp. 184–323.

[3]Bryson, A. E., and Ho, Y. C., *Applied Optimal Control*, Hemisphere, Washington, DC, 1975, pp. 42–127.

[4]Schlemmer, M., and Agrawal, S. K., "Globally Feedback Linearizable Time-Invariant Systems: Optimal Solution Using Mayer's Cost," *Journal of Dynamic Systems, Measurement and Control*, Vol. 120, No. 2, 2000, pp. 343–347.

[5]Fliess, M., Levine, J., Martin, P., and Rouchon, P., "Flatness and Defect of Nonlinear Systems: Introductory Theory and Examples," *International Journal of Control*, Vol. 61, 1995, pp. 1327–1361.

[6]Franch, J., "Flatness, Tangent Systems, and Flat Outputs," Ph.D. Dissertation, Departament de Matematica Aplicada, Technical Univ. of Catalonia, Girona, Spain, Sept. 1999.

[7]Hunt, L. R., Su, R., and Meyer, G., "Design for Multi-Input Nonlinear Systems," *Differential Geometric Control Theory*, edited by R. Brockett, R. Millmann, and H. Sussmann, Birkhäuser, Boston, 1983, pp. 268–298.

[8]Nijmeijer, H., and van der Schaft, A. J., *Nonlinear Dynamical Control Systems*, Springer-Verlag, Berlin, 1990, pp. 176–207.

[9]Charlet, M., Lévine, J., and Marino, R., "On Dynamic Feedback Linearization," *System and Control Letters*, Vol. 13, 1989, pp. 143–151.

[10]Spong, M. W., and Vidyasagar, V., *Robot Dynamics and Control*, Wiley, New York, 1989, pp. 259–273.

Spin-Axis Stabilization of a Rigid Spacecraft Using Two Reaction Wheels

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Introduction

DESIGN of control laws for underactuated spacecraft is important in practice. When one or more actuators fail, the attitude stabilization should be performed by using control laws for underactuated spacecraft with the remaining actuators. The problem of stabilization of a rigid spacecraft using two control inputs has been addressed by several researchers.[1,2] Recently, a discontinuous feedback control law was provided for the stabilization of a spacecraft about any equilibrium attitude.[1] In Ref. 1, it was conjectured without formal proof that the closed-loop system of a spacecraft with the proposed control law might be globally and asymptotically stable. Tsiotras and Longuski presented a new methodology for constructing feedback control laws for the attitude stabilization about the symmetry axis.[2] However, most previous researchers have considered the problem of controlling a spacecraft using less than three thrusters.

On the other hand, it is well known that with less than three momentum wheels the system becomes uncontrollable.[3] When the total angular momentum vector of the spacecraft was assumed to be zero, Krishnan et al. considered the stabilization problem of an underactuated rigid spacecraft using two momentum wheels.[4] In this Note, we consider the problem of spin stabilization of a spacecraft about a specified inertial axis using two reaction wheels. We derive a feedback control law that globally and asymptotically stabilizes the spacecraft about a revolute motion along a specified inertial axis.

Problem Statement

Consider the rotational dynamics of a rigid spacecraft controlled by two reaction wheels. We assume that the body-fixed coordinates are selected to be coincident with the spacecraft principal axes and that the rotation axes of the reaction wheels are the spacecraft principal axes. The dynamics of the spacecraft are given by[5]

$$(I_1-J_a)\dot\omega_1=(I_2-I_3)\omega_2\omega_3+h_2\omega_3-u_1 \tag{1}$$

$$(I_2-J_a)\dot\omega_2=(I_3-I_1)\omega_3\omega_1-h_1\omega_3-u_2 \tag{2}$$

$$I_3\dot\omega_3=(I_1-I_2)\omega_1\omega_2-h_2\omega_1+h_1\omega_2 \tag{3}$$

$$\dot h_1=-J_a\dot\omega_1+u_1 \tag{4}$$

$$\dot h_2=-J_a\dot\omega_2+u_2 \tag{5}$$

where I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft including the moment of inertial contributions of the reaction wheels, J_a is the moment of inertia of each reaction wheel about its spin axis, and ω_1, ω_2, and ω_3 are the components of the angular velocity vector along the principal axes of the body-fixed frame. Also, h_1 and h_2 are relative angular momenta of the reaction wheels with respect to their respective rotational axes, h_i are defined as J_a Ω_i, where Ω_i is the wheel speed with respect to the spacecraft, and u_1 and u_2 are the control torques of the reaction wheels. Without loss of generality, we assume that $I_i > J_a$.

In Ref. 6, Tsiotras and Longuski introduced a new parameterization defining the orientation of rotating rigid bodies. This parameterization is especially useful to the problem of the spin-axis stabilization. According to these results, a transformation from an inertial frame to a body-fixed frame can be achieved by first performing an

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